Discrete Local Volatility for Large Time Steps
(Short Version)

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Abstract

We construct a state-and-time discrete martingale which is calibrated *globally* to a set of given input option prices which may exhibit arbitrage.

We also provide a method to take small steps, fully consistent with the transition kernels of the large steps.

The method's robustness vs. arbitrage violations in the input surface makes our approach particularly suited for computations in stressed scenarios. Indeed, our method of finding a *globally* closest arbitrage-free surface under constraints on implied and local volatility is useful in its own right.

We demonstrate the power of our approach by showing its application to affine dividends calibrated to option prices given by proportional dividends, availability of Likelihood Greeks, and to mean-reverting assets such as VIX. We also comment on how to introduce jumps into our processes.

This is a short version and an extension of our earlier paper [DLV] which might be consulted for further details. Most of the material discussed here was also presented at Global Derivatives 2016 [BG16].

Contents

VERSION HISTORY

- V1.0: initial version
- V1.1: incorporate comments from Derman and Huge.
- V1.2: added example for VIX
- V1.21: minor corrections including reference to [DK94]. Fixed a mistake in (23).
- V1.3: corrections regarding our small step approximation
- V1.4: added a comment on how to impose imperfect correlation between VIX futures.

1 Introduction

In this article, we present a very efficient arbitrage-free discrete space martingale model which is calibrated *globally* to a set of option prices. It can be seen as a discrete version of Dupire's famous local volatility [Du96], but is much more robust vs. arbitrage violations in the input data (such as those arising during Stress calculations). It also provides full access to its transition kernels between each time step, hence it lends itself to numerically efficient calculation methods such as Likelihood Ratio greeks. We provide additional details on the use of jumps, the application to affine dividends and, as new material, how our framework can be applied to mean-reverting assets such as VIX.

Our method is an extension of our earlier work [Bu06], now using the idea of Andreasen and Huge [AH10] to utilize implicit FD operators to construct transition operators, together with a new interpolation scheme for FD grids which ensures that a discrete density remains a density across changing strike grids. We also present a method to take small steps which are fully consistent with the large step transition kernels, if that were required by the payoffs we wish to risk manage.

When compared with related literature, it should be noted that it is *not* the aim of our approach to model an otherwise diffusive process, but to retain the discrete state view of the world throughout. In many ways, our approach is a globally calibrated, implicit version of Derman and Kani's implied trees [DK94].

This article is an abbreviated version of the much more extensive [DLV] which should be consulted for proofs on various statements. It follows the presentation [BG16] we gave at Global Derivatives 2016.

Global Fit

In our view, the strength of our approach is the ability to find a globally closest fit to an input option surface. While global calibration used to be the norm for "structural models" such as time-homogeneous stochastic volatility models a'la Heston's [He93], modern finance prefers "local volatility" overlay models which fit input option surfaces in theory perfectly, but in practise require an unstable local fit or, at best,"boostrapping". As such they are inherently expensive and vulnerable to arbitrage. The method presented here combines the attractiveness of "fitting" with the robustness of "global calibration".

This is the genuinely new contribution of our approach.

Numerical Simplicity

The second advantage of our approach is that it is *very* simple to implement. Every step is explicit: finding a globally best fit uses the simplest of all optimization techniques: linear programming. Once done, assuming a market is arbitrage-free, one may construct the transition operators of our martingales using just simple matrix operations in Excel or in Python. This is in stark contrast to numerically heavy calibration schemes which use higher order mathematics — which, in the end, achieve the same or less robust results. Besides intellectual preference, that means our approach is numerically more robust. Specifically, being able to bound local volatility within the model framework during the "global fit" means that each of the inverse matrices we need to take is also well-defined and bounded.

2 Model Framework

We will model our process as a martingale with unit expectation. The below setup reflects this approach:

Maturities: we are given maturities $0 < t_1 < \cdots < t_m$. We define $dt_j := t_j - t_{j-1}$.

Strikes: for each maturity t_j, we are given strikes $0k_j^{-1} < \cdots 1 \cdots < k_j^{n_j}$ around the unit forward of the asset. We assume that our strike range is increasing, i.e. $k_j^{-1} \leq k_{j-1}^{-1}$ and $k_j^{n_j} \geq k_{j-1}^{n_j-1}$. When modelling non-negative martingales, we will impose $0 \leq k_j^{-1}$. We also set $dk_+^i := k_j^{i+1} - k_j^i$ as well as $dk_-^i := k_j^i - k_j^{i-1}$.

We also define "ghost strikes" $k_j^{-2} := k_j^{-1} - 0.1$ and $k_j^{n_j+1} := k_j^{n_j} + 0.1$ for simplification of notation. Note that k_j^{-2} might be negative.

Densities: we call a non-negative vector p_j a (marginal) density over k_j if it has non-negative weights over strikes $k_j^{-1}, \ldots, k_j^n$ and sums up to one. We denote by $\mathcal{P}_j$ the affine set of densities over our strike grid.

Options: for each strike k_j^i with $j = 0, \ldots, m$ and $i = -2, \ldots, n_j + 1$ we are given call option prices $\mathbb{C}_j^i \equiv \mathbb{E}[(X_{t_j} - k_j^i)^+]$. Model prices will be denoted by $C = (C_j^i)_j^i$. "Admissible" call prices are those which are intrinsic in the boundary strikes $k^{-2}, k^{-1}, k^{n_j}, k^{n_j+1}$. We write $\mathcal{C}_j$ for the affine set of admissible call prices for a specific maturity, and $\mathcal{C}$ for the set of admissible call prices across maturities.

Today: we also add for notational convenience an initial maturity $t_0 := 0$ with $n_0 = 1$ dummy strikes $k_0^i := 1 + 0.1i$ for $i = -2, \ldots, 2$, valued at intrinsic values $C_0^i := (1 - k_0^i)^+$.

Weights: we assume we are given a set of positive weights $w = (w_j^i)$ with $\sum_{i,j} w_j^i = 1$ which define the norm

$$W(C) := \sum_{i,j} w_j^i |C_j^i - \mathbb{C}_j^i| \ . \tag{1}$$

Cash market data: in case "cash" option prices $\mathbb{C}\$$ are given for "cash strikes" K_j^i on an asset with forwards F_j under discount factors DF_j, we may use $\mathbb{C}_j^i := \mathbb{C}\$_j^i/(\mathrm{DF}_j F_j)$ for strikes $k_j^i := K_j^i/F_j$.

This approach is equivalent to modelling a process with discrete but proportional "Black & Scholes" dividends. We will mention in section 4.1 how to incorporate affine dividends and deterministic credit risk a'la [Bu10].

2.1 Linear Algebra

We consider several linear operators over our strikes k_j: for $p_j \in \mathcal{P}_j$ we may define the associated call prices as

$$C_j := J_j^2 p_j \quad \text{with } J_j^2 \text{ defined via} \quad C_j^i := \sum_{u=i+1}^{n} p_j^u (k_j^u - k_j^i) \ . \tag{2}$$

That means J_j^2 is a second order discrete integral operator.[1] Note that $C \in \mathcal{C}_j$. Its inverse[2] over the set of admissible call prices $\mathcal{C}_j$ is denoted by D_j^2, i.e.

$$p_j = D_j^2 C_j \quad \text{with} \quad (D_j^2 C_j)^i = \frac{C^{i+1} - C^i}{dk_+^i} - \frac{C^i - C^{i-1}}{dk_-^i} \ . \tag{3}$$

Hence, it is related to the classic second order difference operator Δ_j^2 by $(D_j^2 C_j)^i = \frac{1}{2}(dk_+^i + dk_-^i)(\Delta_j^2 C_j^i)$.[3] We may use the latter to define Gamma as

$$\Gamma_j := \Delta_j^2 C_j \ . \tag{4}$$

Note that Gamma is the second order derivative with respect to strike, <u>not</u> spot. With the above we also have

$$p_j = \frac{1}{2}(dk_+^i + dk_-^i)\Gamma_j \ . \tag{5}$$

Given a density p_j over k_j, define for arbitrary strikes x the call price function

$$C_j(x) := \sum_u p_j^u (p_j^u - x)^+ . \tag{6}$$

This is equivalent to linear interpolation in the associated calls to p_j. We use it to define (Backward) Theta as

$$b\Theta_j^i := C_j^i - C_{j-1}(k_j^i) \ . \tag{7}$$

Definition 2.1 *Let M be a non-negative matrix. We call M a* probability matrix *if its columns are densities, i.e. $1'M = 1'$. We say M has the* martingale property *from k_1 to k_2 if $k_2'M = k_1'$, in which case we call M a* martingale kernel.

Moreover, let p_{j-1} and p_j be two densities, then M is a transition matrix *if it is a probability matrix with $p_j = Mp_{j-1}$. If it also has the martingale property, we call it a* transition kernel.

The following theorem is crucial in the development of probability matrices. A brief proof is provided in the appendix.

Theorem 2.1 (Construction of Transition Matrices) *Assume that M is a square matrix with columns [rows] add up to 1, and all whose off-diagonal elements are non-positive.*

Then, its inverse exists, is non-negative, and its columns [rows] which add up to 1; in other words M^{-1} is a probability matrix.

Moreover, if M has the martingale property from k_2 to k_1, then M^{-1} has the martingale property from k_1 to k_2.

[1] The equivalent operator in continuous space is $(J^2 f)(x) := \int_x^\infty dy \int_y^\infty dz f(z)$.

[2] For a proof, see proposition A.1 on page 14 in the appendix.

[3] In continuous space, $D^2 \equiv \Delta^2$ and of course $D^2 f(x) := \partial_{xx}^2 f(x)$.

3 Discrete Local Volatility

We wish to find those model option prices $C \in \mathcal{C}$ which are "closest" under W to our market prices $\mathbb{C}$ and which have a *martingale representation* in the sense that there is a sequence of transition kernels $\Pi = (\Pi_j)_j$ such that

$$C_j = J_j^2 p_j \quad \text{with} \quad p_j := \prod_{j:\ell \le j} \Pi_\ell \ .$$

3.1 Absence of Arbitrage for a Single Maturity

For a single maturity j we obviously have:

THEOREM 3.1 (Absence of Arbitrage for a Single Maturity) *Assume that $C_j \in \mathcal{C}_j$ and that $\Gamma_j \ge 0$ (including on the boundary strikes k_j^{-1} and $k_j^{n_j}$).*

Then, $p_j := D_j^2 C_j$ is a density and we call C_j arbitrage-free*.*

3.2 Handling inhomogeneous strikes across time

In this section, we construct at the same point t_{j-1} a transition kernel from one set of strikes to the next; in other words, we define a grid interpolation method which retains the martingale property of the density accross changes ins strikes.. This method is therefore of independent interest for forward-PDE finite difference schemes on time-inhomogeneous grids.

Let $p_{j-1} \in \mathcal{P}_{j-1}$ and define the calls

$$\tilde{C}_{j-1}^i := C_{j-1}(k_j^i) \tag{8}$$

over the strikes of the *next* maturity. This yields the interpolation operator

$$\Xi_j : p_{j-1} \longmapsto \tilde{p}_{j-1} := D_j^2 \tilde{C}_{j-1} \in \mathcal{P}_j \ . \tag{9}$$

THEOREM 3.2 *The operator Ξ_j is a transition kernel.*

This is proven in [DLV].

3.3 Transition Operators from Implicit Schemes

Assume now that $\tilde{p}_{j-1}$ is given, hence we will work on strikes k_j. We use ideas from [AH10] to construct a transition kernel: to this end, we chose a natural "prior" model, $dX_t = \Sigma_t(X_t)\, dW_t$ whose density π satisfies the forward PDE

$$d\pi_t = \frac{1}{2}\partial^2 \left\{ \Sigma_t \pi_t \right\} \, dt \ . \tag{10}$$

This prior model helps us to chose a general structure for our process. We do not actually need the prior process, nor will we later refer back to it.

The implicit FD for (10) is given as

$$p_j - \tilde{p}_{j-1} = \frac{1}{2}\Delta^2 \left\{\Sigma_j \cdot p_j\right\} dt_j = \frac{1}{2} D^2 \left\{\Sigma_j \cdot \Gamma_j\right\} dt_j \ .$$

Here we used $a \cdot b$ to denote element-wise multiplication of two vectors a and b. Applying J^2 yields $C_j - \tilde{C}_{j-1} = \frac{1}{2}\Sigma_j \Gamma_j \, dt_j$, and therefore naturally to defining *Backward Local Volatility* as

$$\Sigma_j^i := \sqrt{\frac{C_j^i - \tilde{C}_{j-1}^i}{\frac{1}{2}\Gamma_j^i dt_j}} \ . \tag{11}$$

Plugging this back into our FD operator gives us

$$I_j^{-1} := 1 - D^2 \left\{ \frac{C_j - \tilde{C}_{j-1}}{p_j} \cdot \circ \right\} \tag{12}$$

such that as promised $I_j^{-1} p_j = \tilde{p}_{j-1}$. We note that the implicit operator is a transition kernel thanks to theorem 2.1. Put together, we get:

Theorem 3.3 (Discrete Local Volatility for Large Steps) *Assume we are given bounds $0 \leq \sigma_- < \sigma_+$ on (backward) local volatility. Let $C \in \mathcal{C}$ satisfy the linear constraints*

$$\left\{ \begin{array}{l} \Gamma_j \geq 0 \ , \\ \frac{1}{2}\Gamma_j dt_j \ k_j^2 \sigma_-^2 \leq \not{b}\Theta_j \leq \frac{1}{2}\Gamma_j dt_j \ k_j^2 \sigma_+^2 \ . \end{array} \right. \tag{13}$$

Then C is arbitrage-free with representation $\Pi = (\Pi_j)_j$ given by

$$\Pi_j := I_j \Xi_j \ . \tag{14}$$

Its "Backward Local Volatility" Σ is bounded,

$$k_j^2 \sigma_-^2 \leq \Sigma_j \leq k_j^2 \sigma_+^2 \ . \tag{15}$$

Note that Σ is not actually used in the construction of Π. It is only needed in order to be able to specify intuitive bounds on $\not{b}\Theta$ via (15). Those bounds make sure that (12) remains bounded which drastically reduces numerical errors when Γ and therefore p is close to zero.

Remark 3.1 *Note that positivity of "Forward Theta" $C_{j+1}(k_j^i) - C_j^i$ on a time-inhomogeneous grid is* not *sufficient to ensure absence of arbitrage, c.f. [DLV].*

3.4 Small Steps

Assume now that C is arbitrage-free with martingale representation (14), and that we want to create transition matrices $(\hat{\Pi}_k)_{k=0}^M$ for intermediate time steps $t_{j-1} = \tau_0 < \cdots < \tau_M = t_j$.

To this end, we eigen-decompose the diagonalizable matrix I_j into

$$I_j = U_j D_j U_j^{-1}$$

where D_j is a diagonal matrix whose diagonal elements are the eigenvalues of I_j; c.f. [DLV] for why this is possible. We then define

$$\hat{\Pi}_k := U_j D^{\frac{\tau_k - \tau_{k-1}}{t_j - t_{j-1}}} U_j^{-1} \tag{16}$$

for $k = 1, \ldots, M$ and additionally $\hat{\Pi}_0 := \Xi_j$.

THEOREM 3.4 *Each $\hat{\Pi}_k$ is a transition kernel.*

This was proven in [DLV]. Evidently,

$$\Pi_j = \prod_{k=0}^{M} \hat{\Pi}_k \ , \tag{17}$$

which means that our small steps are consistent with our large steps.

3.5 Summary

The above results can be summarized as follows:

1. Find the W-closest surface $C \in \mathcal{C}$ which satisfies the linear conditions (13).
2. Construct its martingale representation Π using (12).
3. Use (16) to construct small steps between large steps where needed.

3.6 Jumps

A simple extension of our approach is the incorporation of conditionally independent jumps. The basic idea is to write our martingale for each period as a product of a jump process and our discrete "local volatility" martingale. During simulation, that means the asset will first jump according to the jump process, and then continue on given by the above introduced "diffusive" approach.

Assume that $U_j(x)$ is an independent non-negative bounded random variable, parameterized by $x \in k_{j-1}$, such that $\mathbb{E}[U_j(x)] = 1$ for all x and therefore that $\mathbb{E}[Z_{j-1}U_j(Z_{j-1})|Z_{j-1}] = Z_{j-1}$ for any random variable Z_{j-1} over k_{j-1}.

EXAMPLE 1 *Assume $N(x)$ is a Poission-processes with intensity $\lambda(x)$, and that $(\xi_t)_t$ is a sequence of relative iid jumps which are bounded from above. Let $\Lambda(x) := \mathbb{E}[e^{\xi_1} - 1]\lambda(x)$ and define*

$$U_j(x) := \exp\left\{ \int_{t_{j-1}}^{t_j} \xi_s \, dN_j(x) - \Lambda(x) dt_j \right\} \tag{18}$$

such that indeed $\mathbb{E}U_j = 1$, $U_j \geq 0$ and $\sup U_j < \infty$. A good example are "Stress scenario" jumps of a fixed size with a "local" frequency which depends on the level of the spot.

In order to apply our jumps to our framework, we "convolute" our discrete martingale with our jumps as follows: instead of using (8) to define $\tilde{C}$, let

$$\tilde{C}_j^i := \sum_{r=-1}^{n_{j-1}} p_{j-1}^r \mathbb{E}\left[\left(U_j(k_{j-1}^r)\, k_{j-1}^r - k_j^i \right)^+ \right] \tag{19}$$

which is again a linear operator of p_{j-1}. Note, however, that the call prices will not have intrinsic value in k_j^{-1} and $k_j^{n_j}$ unless the grid widens sufficiently from t_{j-1} to t_j. This is captured in the following statement:

Theorem 3.5 *If $k_j^{-1} \leq \inf U_j\, k_{j-1}^{-1}$ and $k_j^{n_j} \geq \sup U_j\, k_{j-1}^{n_{j-1}}$, then $C_j \in \mathcal{C}_j$.*

If the conditions in our theorem apply, we may proceed to re-define the density

$$\tilde{p}_j := D_j^2 \tilde{C}_j \in \mathcal{P}_j \ .$$

As before, the operator $\Xi_j : p_{j-1} \mapsto \tilde{p}_j$ is a martingale kernel. All the previous results hold when using this new "interpolation method" from k_{j-1} to k_j.

Most notably, our approach with jumps allows us to find a globally closest set of option prices closest to the input prices which are compatible with our jump assumptions and satisfy reasonable bounds on (backward) local volatility.

4 Applications

We briefly mention three specific applications which demonstrate the power of our approach: pricing with affine dividends when the input surface is given with proportional "Black & Scholes" dividends, use of Likelihood Greek ratio risk, and, finally, an application for pricing mean-reverting assets such as VIX.

4.1 Affine Dividends and Deterministic Credit Risk

A nice application of our method of finding a closest arbitrage-free model with bounded local volatility is the usage of affine dividends in a context where the market prices were marked with classic proportional "Black & Scholes" dividends, and without any reference to default risk.

We first introduced the affine dividend framework in [Bu10]: assume that we wish to model a dividend paying stock price process S which drifts at a funding rate μ and pays at ex-dividend dates $0 < \tau_1 < \cdots$ affine dividends $\Delta_{\tau_k} = \alpha_k + S_{\tau_k -}\beta_k$. Assume further that the asset is subject to deterministic credit risk in the sense that there exists an independent exponentially distributed default time τ with deterministic intensity $h \geq 0$ such that $\mathbb{S}_t := \mathbb{E}[S_t > 0] = \mathbb{P}[\tau > t] = \exp\{-\int_0^t h_s\, ds\}$. Obviously, once the stock has defaulted, no further dividends are paid. We also assume that the risk-free discount factor is given by DF_t.

We write the stock price process as

$$S_t = S_t^* 1_{\tau > t} \ ,$$

where S^* is the stock price process conditional on survival.

We transform β into discrete yields $d_k := -\log(1-\beta_k)$ such that $\beta_k := 1-\exp(-d_k)$. The forward of S^* is then given as

$$F_t^* = R_t^* \left(S_0 - \sum_{k:\tau_k \le t} \frac{\alpha_k}{R_{\tau_k}^*} \right) \quad \text{with} \quad R_t^* := \exp\left\{ \int_0^t (h_s + \mu_s)\, ds - \sum_{\ell:\tau_\ell \le t} d_k \right\} \ .$$

We have shown in [Bu10] that if S^* is to remain positive, then there *always* exists a "pure" non-negative martingale X such that

$$S_t^* = (F_t^* - D_t^*) X_t + D_t^*$$

where D^* represents the future discounted dividends,

$$D_t^* := R_t^* \sum_{k:t<\tau_k} \frac{\alpha_k}{R_{\tau_k -}^*} \ .$$

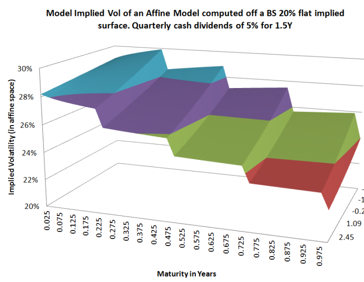
Assume now that we observe market option prices $\mathbb{C}\$_j^i := \mathrm{DF}_j\, \mathbb{E}\left[\, (S_j - K_j^i)^+ \,\right]$ on S for cash strikes K_j^i. Define relative strikes $k_j^i := (K_j^i - D_j^*)/(F_j^* - D_j^*)$ such that option prices on X are given by

$$\mathbb{C}_j^i := \frac{\mathbb{C}\$_j^i}{\mathrm{DF}_j \mathbb{S}_j (F_j^* - D_j^*)} \ .$$

Then, our results can be applied directly to $\mathbb{C}$, which means that we have provided a methodology to find option prices under cash dividends and credit risk, globally closest to a surface of option prices calibrated to proportional dividends without credit risk. This methodology completes the results presented in [Bu10].

Example Affine Dividends — we consider a case of a stock without credit risk which we predict will pay, rather generously, 5% dividends quarterly for the next six quarters. We assume that the option prices are marked using Black & Scholes proportional dividends at 20% flat volatility.

Such an option price surface is not arbitrage-free for a cash-dividend process as can be seen on the left hand graph in figure 4.1. The right hand graph shows the closest arbitrage-free fit with heavy weights on the option prices at the dividend dates.



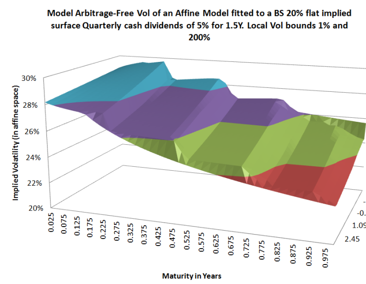
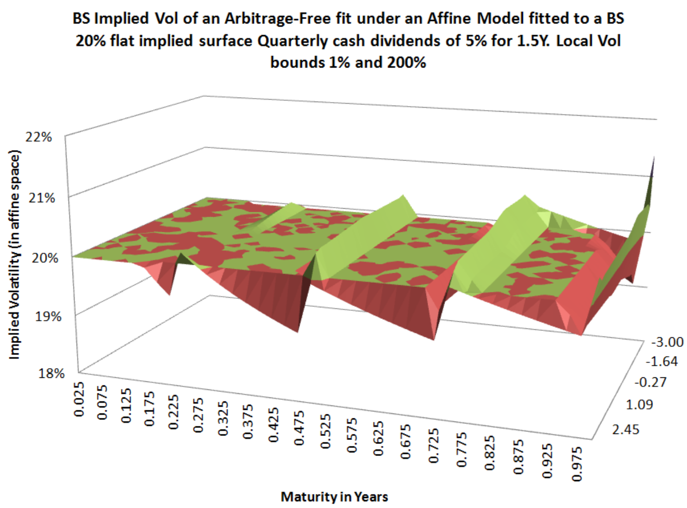


Figure 4.1 in turn shows the resulting implied volatility, mapped back into the Black&Scholes proportional dividend case.

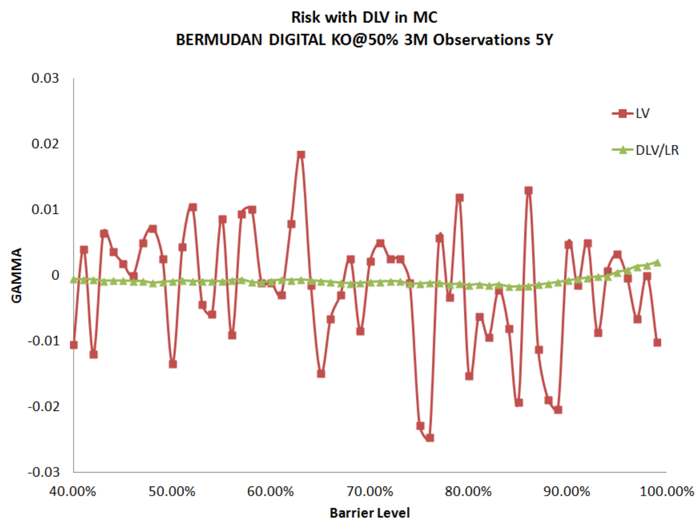


4.2 Likelihood Ratio Greeks

In this brief section, we point out that since in our model we have access to the explicit transition operators, the model lends itself to Likelihood Ratio greek calculations. Assume, for example, that $\Pi^* = (\Pi_j^*)_j$ is the martingale representation for our model under a perturbation of one of the market inputs such as volatility for calculation of Vega. To compute the "perturbed" price of a payoff H, we may simply evaluate $\mathbb{E}[H^*(X_1, \ldots, X_m)]$ with H^* defined as

$$H^*(x_1, \ldots, x_m) := H(x_1, \ldots, x_m) \prod_{j=1}^{m} \frac{\Pi_j^*(x_j|x_{j-1})}{\Pi_j(x_j|x_{j-1})} .$$

Figure 4.2 illustrates the power of this method when computing the Gamma (second derivative in spot) as a function of different levels of moneyness for a Bermudan Digital with 5Y maturity and 3M observations.



Of course, care needs to be taken to ensure that a perturbation does not violate arbitrage, in particular for strike/maturity-wise vega. This is easily achieved by a localized version of our "fitting" problem which ensures that only the intended options prices are being moved by the fitting program.

4.3 Pricing Mean-Reverting Assets: VIX

As a final application, we will show how our approach can be used to develop pricing models for underlyings which are commonly modelled as mean-reverting assets. The example we have in mind is spot-VIX which we wish to model across listed options on VIX futures. We basically present a discrete state version of one of the models proposed by Drimus and Farkas in [DF13]. Our version has the advantage of providing fast "closed form" access to VIX futures of all maturities at all intermediate times. That said, we will not comment here on how to model a consistent equity process; this will be subject to a further paper.

Of course, our approach applies as well to similar underlyings such as commodity futures for which only terminal options per future are available.

As before, we assume that we are given all option prices relative to the forward of the underlying process. The forwards here are given by the futures, which means that all strikes are relative to the futures level of the VIX at their respective maturity,

Following [DF13] we assume (spot-) VIX is a mean-reverting process. This leads to the following "prior" model: let Z be the martngale

$$dZ_t = \Sigma_t(Z_t)\, dW_t \ , \quad Z_0 := 1 \tag{20}$$

and define for $\kappa \geq 0$ the VIX process

$$X_t := 1 + e^{-\kappa t} (Z_t - 1) \ . \tag{21}$$

Clearly, $\mathbb{E}[X_t] = 1$. Then, X solves

$$dX_t = \kappa\,(1 - X_t)\,dt + \Sigma_t^X(X_t)\,dW_t \tag{22}$$

with $\Sigma_t^X(x) := e^{-\kappa t}\Sigma_t\,(e^{\kappa t}(x-1)+1)$.

An option on the t_j-VIX future at strike k_j^i in our notation is written as

$$\bar{\mathbb{C}}_j^i \equiv \mathbb{E}\left[\,\left(1 + e^{-\kappa t_j}\,(Z_j - 1) - k_j^i\right)^+\,\right]\ .$$

We transform these into options $\mathbb{C}_j^i := e^{-\kappa t_j}\bar{\mathbb{C}}_j^i$ on Z over the grid of strikes

$$z_j^i := e^{\kappa t_j}\left(k_j^i - 1\right) + 1\ .$$

This allows us applying our Discrete Local Volatility approach to the martingale Z with unit expectation. Note that Z is not guaranteed to remain positive, hence we may find $z_j^{-1} < 0$.

The closest call price surface C with martingale representation $\Pi = (\Pi_j)_j$ yields via the transformation (21) the desired VIX process for X. It will remain non-negative simply because the process in our methodology can only take values at the strikes provided for options on VIX, which are naturally all positive.

VIX Futures, any time

Assume now that we wish to compute the value of the VIX future maturing at t_j at an earlier time $t \in (t_{\ell-1}, t_\ell]$, $\ell \le j$. We can compute the respective transition matrix $\mathbb{P}[Z_{t_j}|Z_t]$ for Z using our previous notation as

$$\Pi_{j|t} := U_\ell' D_\ell^{\frac{t_\ell - t}{dt_\ell}}\,U_\ell \cdot \prod_{u=\ell+1}^{j} \Pi_u \tag{23}$$

Hence, we may compute the value of the t_j-future at t as the conditional expectation

$$X_j(t) = 1 + e^{-\kappa t_j}\left(z_j'\Pi_{j|t} - 1\right)\ .$$

This is a very efficient calculation in both MC and FD schemes. Prices of options on VIX futures can be computed in a similar way at any point prior to their expiry.

Remark 4.1 *In the above one-factor approach, all dividend futures have perfect correlation. If one wishes to introduce imperfect correlation, we may define separate processes $X^{(j)}$ per maturity, each using the same previously calibrated transition operators. Samples in a Monte-Carlo for each time step are then drawn from a suitably estimated copula.*

5 Summary

In this article, we have presented with our "Discrete Local Volatility" the first local-volatility-style model which allows global fitting to a set of possibly arbitrage-free input surfaces. Jumps are naturally supported.

Since our approach provides explicit access to the transition operators even for large steps, it is numerically very efficient and allows the use of various advanced numerical schemes, for example Likelihood Greeks. We have also shown how the method can be extended to mean-reverting assets such as VIX.

A Proofs

PROPOSITION A.1
For $C \in \mathcal{C}_j$ we have $J_j^2 D_j^2 C = C$. For $p \in \mathcal{P}_j$, we have $D^2 J^2 p = p$.

Proof – We drop the index j. We show first $J_j^2 D_j^2 C = C$:

$$
\begin{aligned}
J_j^2 D_j^2 C &= \sum_{u=i+1}^{n} (k^u - k^i) \left\{ \frac{C^{u+1} - C^u}{dk_+^u} - \frac{C^u - C^{u-1}}{dk_-^u} \right\} \\
&\overset{(C^n = C^{n-1} = 0)}{=} \sum_{u=i+1}^{n-1} (k^u - k^i) \left\{ \frac{C^{u+1} - C^u}{dk_-^{u+1}} \right\} - \sum_{u=i}^{n-1} (k^{u+1} - k^i) \left\{ \frac{C^{u+1} - C^u}{dk_-^{u+1}} \right\} \\
&= \sum_{u=i+1}^{n-1} -dk_-^{u+1} \frac{C^{u+1} - C^u}{dk_-^{u+1}} - (k^{i+1} - k^i) \frac{C^{i+1} - C^i}{dk_-^{i+1}} \\
&= -C^i + C^{i+1} - C^n + C^{i+1} \;=\; C^i \ .
\end{aligned}
$$

To show $D^2 J^2 p = p$ note first

$$
\begin{aligned}
(J^2 p)^{i+1} - (J^2 p)^i &= \sum_{u=i+2}^{n} p^u (k^u - k^{i+1}) - \sum_{u=i+1}^{n} p^u (k^u - k^i) \\
&= -p^{i+1}(k^{i+1} - k^i) - (k^{i+1} - k^i) \sum_{u=i+2}^{n} p^u \\
&= -dk_+^i \sum_{u=i+1}^{n} p^u
\end{aligned}
$$

Hence,

$$
D^2 J^2 p = \frac{-dk_+^i \sum_{u=i+1}^n p^u}{dk_+^i} - \frac{-dk_+^{i-1} \sum_{u=i}^n p^u}{dk_+^{i-1}} = p^i \ .
$$

□

Proof of theorem 2.1 on page 5– A matrix M with the referenced properties is a Z-matrix, i.e. $M^{i,i} > 0$ and $M^{i,\ell} \leq 0$ for $i \neq \ell$. Moreover, its columns add up to

1, which means it is a non-singular M-matrix, c.f. [BR94], chapter 6: equivalent classification I_{29} of non-singular M-Matrices.[4]

Hence, its inverse exists and is non-negative, c.f. [BR94], chapter 6: equivalent classification N_{38} of non-singular M-Matrices.

Finally, $1'M = 1'$ implies $1' = 1'M^{-1}$, i.e. the columns of M^{-1} add up to 1, too. Similarly, if $k_2'M = k_1'$, then $k_2' = k_1'M^{-1}$. $\square$

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